

# Filters in Fuzzy Class Theory

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## Abstract

The notion of filter is established in fuzzy class theory. Graded properties of filters, prime filters and related concepts are investigated. In particular we study properties of filters relativized with respect to the basic three t-norms: Gödel, product, and Łukasiewicz. It is shown that our approach gives rise to a broad class of models including probability and necessity measures on fuzzy sets as special cases.

*Keywords:* filter, prime filter, fuzzy class theory, logic  $\mathbb{L}\Pi$

## 1 Introduction

A notion of filter as a particular family of subsets of a given universe was established by F. Riesz at the beginning of the last century and further developed by A. Tarski and H. Cartan. Nowadays filters are tools of extreme importance in many areas of classical mathematics. For example, in topology they enhance the concept of convergence and, in measure theory, prime filters can be interpreted as basic components of probability measures.

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In fuzzy mathematics filters have been conceived in various manners. Let us briefly mention some of the existing approaches in no particular order. Prefilters as crisp sets of fuzzy sets were introduced by Lowen [15, 16] whose aim was to develop a theory of fuzzy uniform spaces. Burton et al. [4, 5] came with an idea of a so called generalized filter as a certain fuzzy set of crisp sets which turned out to be useful in the context of generalized uniform spaces. From the viewpoint of Hájek's style mathematical fuzzy logic [11], connectives and operations used in both concepts correspond to Gödel logic; that is, conjunction and disjunction are interpreted by minimum and maximum, respectively (for intersection and union accordingly). More general filter theory was proposed by Höhle and Šostak [14] who introduced a so called  $L$ -filter as a certain fuzzy set of fuzzy sets. Filters with values in the standard algebra of Łukasiewicz logic (MV-algebra) were considered by Höhle [13]. A detailed discussion of various notions of filters can be found in [10].

The primary aim of this contribution is to introduce a sufficiently general filter theory based on graded definitions. We study its properties within the formal framework of *fuzzy class theory* which was proposed in [2] as a foundational theory for fuzzy mathematics. We restrict ourselves to basic notions and results related to properties of filters, bases, prime filters, and elementary constructions. Most concepts have already appeared in the literature, nevertheless the approach presented herein is more general (we can work almost in an arbitrary fuzzy logic) and more expressive (we investigate *graded* properties of fuzzy filters). We shall see that our approach is an appropriate generalization of classical filter theory: interestingly, our proofs are in most cases just an adaptation of classical ones into the scope of fuzzy logic. See [1] for a detailed treatment of related methodological issues.

The paper is structured as follows. Section 2 contains necessary preliminary notions: summary of the fuzzy logic used and fuzzy class theory. Section 3 is entirely devoted to the study of graded properties of filters and prime filters. Models arising from the investigation of filters in our formal framework are discussed in Section 4 and their relation to the existing notions of filters is discussed. Section 5 contains concluding remarks.

## 2 Preliminary Notions

### 2.1 ŁŁ Logic

There are three basic fuzzy logics covered in the general Hájek's approach [11] to mathematical fuzzy logic: *Gödel*, *product*, and *Łukasiewicz logic*. There were proposed several logical systems incorporating properties of all basic fuzzy logics, which offers huge expressive power. In this paper, we use logic ŁŁ [9]. This logic can be described (roughly speaking) as a fuzzy logic of continuous t-norms representable as finite ordinal sums of Gödel, product, and Łukasiewicz t-norms: in other words, every conjunction interpreted by such a t-norm is definable within logic ŁŁ [7]. The standard algebra of truth values is the real interval  $[0, 1]$ . We are not going to repeat the definition and basic properties of logic ŁŁ and its first-order predicate version — the reader is referred to [9, 7] for comprehensive details. In the sequel, we shall only recall notation and basic definitions.

The logical connectives below can be defined in ŁŁ; they are listed together with their standard semantics in  $[0, 1]$  and we use the same symbols for logical connectives as for the corresponding algebraic operations:

0	0	truth constant falsum
1	1	truth constant verum
$\neg_L \varphi$	$\neg_L x = 1 - x$	involutive negation
$\neg_P \varphi$	$\neg_P x = 1 - \text{sgn}(x)$	strict (product) negation
$\varphi \wedge \psi$	$x \wedge y = \min(x, y)$	min-conjunction
$\varphi \&_P \psi$	$x \&_P y = x \cdot y$	product conjunction
$\varphi \&_L \psi$	$x \&_L y = \max(0, x + y - 1)$	Łukasiewicz conjunction
$\varphi \vee \psi$	$x \vee y = \max(x, y)$	max-disjunction
$\varphi \underline{\vee}_P \psi$	$x \underline{\vee}_P y = x + y - x \cdot y$	product disjunction
$\varphi \oplus \psi$	$x \oplus y = \min(1, x + y)$	Łukasiewicz disjunction
$\varphi \rightarrow_G \psi$	$x \rightarrow_G y = \begin{cases} 1, & \text{if } x \leq y \\ y, & \text{otherwise} \end{cases}$	Gödel implication
$\varphi \rightarrow_P \psi$	$x \rightarrow_P y = \begin{cases} 1, & \text{if } x \leq y \\ \frac{y}{x}, & \text{otherwise} \end{cases}$	product implication
$\varphi \rightarrow_L \psi$	$x \rightarrow_L y = \min(1, 1 - x + y)$	Łukasiewicz implication
$\Delta \varphi$	$\Delta x = \begin{cases} 1, & \text{if } x = 1 \\ 0, & \text{otherwise} \end{cases}$	Baaz delta

Recall that involutive negation  $\neg_L$  corresponds to Łukasiewicz logic and strict negation  $\neg_P$  to both product and Gödel logic.

**Convention 2.1** *Symbols  $\neg, \underline{\vee}, \&, \rightarrow, \leftrightarrow$  stand for an arbitrary negation, disjunction, conjunction, implication, and equivalence, respectively, which is definable in logic  $\mathbb{L}\Pi$ . We always assume that all the connectives without indices appearing in a formula are related to only one  $t$ -norm definable in  $\mathbb{L}\Pi$  (for example, the formula  $\neg\varphi \& (\psi \rightarrow \chi)$  related to Gödel  $t$ -norm reads as  $\neg_P\varphi \wedge (\psi \rightarrow_G \chi)$  and analogously for other  $t$ -norms).*

Recall that equivalence  $\varphi \leftrightarrow \psi$  is an abbreviation for  $(\varphi \rightarrow \psi) \& (\psi \rightarrow \varphi)$ .

## 2.2 Fuzzy Class Theory

We summarize basic definitions of fuzzy class theory. Fuzzy class is formal counterpart of fuzzy set. Recall that from the viewpoint of formal logic, fuzzy class theory can be characterized as Henkin-style higher-order fuzzy logic. For technical details and in-depth discussion of the methodology see [1, 2]. The webpage <http://www.cs.cas.cz/hp/> is completely devoted to fuzzy class theory and results achieved within its framework.

**Definition 2.2 (Henkin-style second-order fuzzy logic)** *Henkin-style second-order fuzzy logic over  $\mathbb{L}\Pi$  is a theory in multi-sorted first-order predicate fuzzy logic  $\mathbb{L}\Pi$  with sorts for atomic objects (lowercase variables  $x, y, \dots$ ) and classes (uppercase variables  $A, B, \dots$ ).*

*Besides the logical predicate of identity  $=$ , the only primitive predicate is the membership predicate  $\in$  between objects and classes. The axioms for  $\in$  are the following:*

1. *The comprehension axiom  $(\exists X)\Delta(\forall x)(x \in X \leftrightarrow \varphi)$ ,  $\varphi$  not containing  $X$ , which enable the (eliminable) introduction of comprehension terms  $\{x \mid \varphi\}$  with axioms  $y \in \{x \mid \varphi(x)\} \leftrightarrow \varphi(y)$  (where  $\varphi$  may be allowed to contain other variables and comprehension terms).*
2. *The extensionality axiom  $(\forall x)\Delta(x \in X \leftrightarrow x \in Y) \rightarrow X = Y$ .*

**Definition 2.3 (Henkin-style higher-order fuzzy logic)** *Henkin-style fuzzy logic of higher orders is obtained by repeating the previous definition on each level of the type hierarchy. Obviously, defined symbols of any type can then be*

shifted to all higher types as well. (Consequently, all theorems are preserved by uniform upward type-shifts.) Types may be allowed to subsume all lower types. Calligraphic letters  $\mathcal{F}, \mathcal{G}, \dots$  denote classes of classes.

Alternatively, the whole theory is called *fuzzy class theory* (over  $\mathbb{LII}$ ). Notice that, despite the name “higher-order logic”, its Henkin-style variant is a theory over first-order logic  $\mathbb{LII}$ .

**Convention 2.4** *The formulas  $\varphi \ \& \ \dots \ \& \ \varphi$  ( $n$  times) are abbreviated  $\varphi^n$ . Furthermore,  $x \notin X$  is a shorthand for  $\neg(x \in X)$ . We use the symbol  $\notin_P$  and  $\notin_L$  in case of strict and involutive negation, respectively. Consecutive use of general quantifier  $(\forall A)(\forall B)$  and existential quantifier  $(\exists A)(\exists B)$  is abbreviated  $(\forall A, B)$  and  $(\exists A, B)$ , respectively.*

**Definition 2.5 (Fuzzy class operations)** *The following elementary fuzzy class operations can be defined:*

$\emptyset$	=df	$\{x \mid 0\}$		<i>empty class</i>
$\mathbb{V}$	=df	$\{x \mid 1\}$		<i>universal class</i>
$\setminus X$	=df	$\{x \mid x \notin X\}$		<i>complement</i>
$\setminus_P X$	=df	$\{x \mid x \notin_P X\}$		<i>strict complement</i>
$\setminus_L X$	=df	$\{x \mid x \notin_L X\}$		<i>involutive complement</i>
$X \cap Y$	=df	$\{x \mid x \in X \ \& \ x \in Y\}$		<i>intersection</i>
$X \sqcap Y$	=df	$\{x \mid x \in X \ \wedge \ x \in Y\}$		<i>min-intersection</i>
$X \cup Y$	=df	$\{x \mid x \in X \ \vee \ x \in Y\}$		<i>union</i>
$X \sqcup Y$	=df	$\{x \mid x \in X \ \vee \ x \in Y\}$		<i>max-union</i>

In addition, we can introduce union and intersection of an arbitrary fuzzy class of fuzzy classes.

**Definition 2.6** *The union and intersection of a fuzzy class  $\mathbb{F}$  of fuzzy classes are defined as follows:*

$$\begin{aligned} \bigcup \mathbb{F} &=_{\text{df}} \{A \mid (\exists \mathcal{F})(\mathcal{F} \in \mathbb{F} \ \& \ A \in \mathcal{F})\} \\ \bigcap \mathbb{F} &=_{\text{df}} \{A \mid (\forall \mathcal{F})(\mathcal{F} \in \mathbb{F} \ \rightarrow \ A \in \mathcal{F})\} \end{aligned}$$

**Definition 2.7 (Fuzzy class predicates and relations)** *Further we define the following predicates and elementary relations between fuzzy classes:*

Height( $X$ )	$\equiv_{\text{df}}$	$(\exists x)(x \in X)$	<i>height</i>
Crisp( $X$ )	$\equiv_{\text{df}}$	$(\forall x)\Delta(x \in X \vee x \notin X)$	<i>crispness</i>
$X \subseteq Y$	$\equiv_{\text{df}}$	$(\forall x)(x \in X \rightarrow x \in Y)$	<i>inclusion</i>
$X \sqsubseteq Y$	$\equiv_{\text{df}}$	$(\forall x)(x \in X \rightarrow_G x \in Y)$	<i>Gödel inclusion</i>
$X \approx Y$	$\equiv_{\text{df}}$	$(\forall x)(x \in X \leftrightarrow x \in Y)$	<i>bi-inclusion</i>

Throughout the paper, we shall use all elementary theorems on the above defined notions which follow from the metatheorems proved in [2], and thus can be checked by simple propositional calculations.

### 3 Graded Properties of Filters

In classical mathematics, a *filter* is a family  $\mathcal{F}$  of subsets of a set  $X$  such that:

1.  $X \in \mathcal{F}$ ,
2.  $\emptyset \notin \mathcal{F}$ ,
3. if  $A \in \mathcal{F}$  and  $A \subseteq B$ , then  $B \in \mathcal{F}$ ,
4. if  $A, B \in \mathcal{F}$ , then  $A \cap B \in \mathcal{F}$ .

Sometimes filter is defined without requiring condition 2. and called *proper filter* when 2. is satisfied. See, for example, [6] for a detailed exposition on filters, their bases and other notions with which we are going to deal in next sections.

#### 3.1 Filters

When we follow the analogy with classical notions faithfully, we conceive filter in fuzzy mathematics as a particular fuzzy class  $\mathcal{F}$  of fuzzy classes (denoted  $A, B, \dots$ ). At first, we define two unary predicates:

$$\begin{aligned} \text{Mon}(\mathcal{F}) &\equiv_{\text{df}} (\forall A, B)((A \in \mathcal{F} \ \& \ A \subseteq B) \rightarrow B \in \mathcal{F}) \\ \text{IntClosed}(\mathcal{F}) &\equiv_{\text{df}} (\forall A, B)((A \in \mathcal{F} \ \& \ B \in \mathcal{F}) \rightarrow A \cap B \in \mathcal{F}) \end{aligned}$$

The predicate  $\text{Mon}(\mathcal{F})$  determines a degree to which a fuzzy class  $\mathcal{F}$  is *monotone*, while the predicate  $\text{IntClosed}(\mathcal{F})$  determines a degree to which  $\mathcal{F}$  is

*closed with respect to intersection.* Realize that truth degrees of both predicates strongly depend on the connectives and operations used. The next lemma summarizes basic properties of a monotone and intersection-closed fuzzy class of fuzzy classes, respectively. We omit general quantifiers whenever possible since the formulas  $\varphi$  and  $(\forall x)\varphi$  are equi-provable.

**Lemma 3.1** *The following formulas are provable:*

1.  $\text{Mon}(\mathcal{F}) \rightarrow (A \cap B \in \mathcal{F} \rightarrow (A \in \mathcal{F} \wedge B \in \mathcal{F}))$
2.  $\text{Mon}^2(\mathcal{F}) \rightarrow ((A \cap B \in \mathcal{F})^2 \rightarrow (A \in \mathcal{F} \& B \in \mathcal{F}))$
3.  $\text{Mon}^2(\mathcal{F}) \rightarrow ((A \in \mathcal{F} \vee B \in \mathcal{F}) \rightarrow A \cup B \in \mathcal{F})$
4.  $\text{Mon}(\mathcal{F}) \rightarrow (B \notin \mathcal{F} \rightarrow (A \notin \mathcal{F} \underline{\vee} A \not\subseteq B))$
5.  $\text{Mon}(\mathcal{F}) \rightarrow ((B \notin \mathcal{F} \& A \subseteq B) \rightarrow A \notin \mathcal{F})$
6.  $\text{IntClosed}(\mathcal{F}) \rightarrow (A \cap B \notin \mathcal{F} \rightarrow \neg(A \in \mathcal{F} \& B \in \mathcal{F}))$
7.  $(\text{IntClosed}(\mathcal{F}) \& \emptyset \notin \mathcal{F}) \rightarrow (A \cap B \approx \emptyset \rightarrow \neg(A \in \mathcal{F} \& B \in \mathcal{F}))$
8.  $(\text{IntClosed}(\mathcal{F}) \& \emptyset \notin \mathcal{F}) \rightarrow (A \cap B \approx \emptyset \rightarrow (A \notin \mathcal{F} \underline{\vee} B \notin \mathcal{F}))$
9.  $(\text{IntClosed}(\mathcal{F}) \& \emptyset \notin \mathcal{F}) \rightarrow (A \notin \mathcal{F} \underline{\vee} \setminus A \notin \mathcal{F})$
10.  $(\text{IntClosed}(\mathcal{F}) \& \emptyset \notin \mathcal{F}) \rightarrow (A \cap B \approx \emptyset \rightarrow (A \in \mathcal{F} \rightarrow B \notin \mathcal{F}))$
11.  $(\text{IntClosed}(\mathcal{F}) \& \emptyset \notin \mathcal{F}) \rightarrow (A \in \mathcal{F} \rightarrow \setminus A \notin \mathcal{F})$

We make two important points here concerning interpretation of formulas within fuzzy logic. First, it is immaterial which implication is used as the main connective in the theorem since we are usually concerned with provability to degree one. Second, notice that the formula in 2. has an antecedent in which the same subformula  $\text{Mon}(\mathcal{F})$  appears twice connected by conjunction. Double occurrence of that formula cannot be omitted unless Gödel conjunction is used. However, adding Baaz delta  $\Delta$  before the assumption enables to avoid double usage of the same assumption: the formula in 2. would then be

$$\Delta \text{Mon}(\mathcal{F}) \rightarrow (\Delta(A \cap B \in \mathcal{F}) \rightarrow (A \in \mathcal{F} \& B \in \mathcal{F}))$$

Concerning the proof of Lemma 3.1, we are following the instructions from [3], where an in-depth discussion of issues related to proving within fuzzy

logic appears: we carry out all proofs in a semi-formal way since their purely formal counterparts can be easily reconstructed. In the proof of Lemma 3.1 (as well as other propositions in this section) we freely use theorems of LII and its three significant fragments (Gödel, product and Łukasiewicz logic).

**Proof:**

1. Assumptions are  $\text{Mon}(\mathcal{F})$  and  $A \cap B \in \mathcal{F}$ . We get  $A \in \mathcal{F}$  from  $\text{Mon}(\mathcal{F})$  and, using the same two premises once again with  $\wedge$ , the result is  $A \in \mathcal{F} \wedge B \in \mathcal{F}$ .
2.  $\text{Mon}^2(\mathcal{F})$  gives together with the second assumption ( $A \cap B \in \mathcal{F}$ )<sup>2</sup> the formula  $A \in \mathcal{F} \& B \in \mathcal{F}$  by double use of  $\text{Mon}(\mathcal{F})$  together with  $A \cap B \in \mathcal{F}$ .
3. Employing  $\text{Mon}(\mathcal{F})$  twice, we derive  $A \in \mathcal{F} \rightarrow A \cup B \in \mathcal{F}$  and the formula  $B \in \mathcal{F} \rightarrow A \cup B \in \mathcal{F}$ , which implies the result.
4. The result follows from the premise  $\text{Mon}(\mathcal{F})$  and two provable formulas of LII:  $(\varphi \rightarrow \psi) \rightarrow (\neg\psi \rightarrow \neg\varphi)$  and  $\neg(\varphi \& \psi) \rightarrow (\neg\varphi \vee \neg\psi)$ .
5. From  $\text{Mon}(\mathcal{F})$  we get  $A \subseteq B \rightarrow (A \in \mathcal{F} \rightarrow B \in \mathcal{F})$  by residuation. We derive  $A \subseteq B \rightarrow (B \notin \mathcal{F} \rightarrow A \notin \mathcal{F})$  from the last formula and the conclusion follows by residuation.
6. Immediately follows from  $\text{IntClosed}(\mathcal{F})$ .
7. The assumptions are  $\text{IntClosed}(\mathcal{F})$ ,  $\emptyset \notin \mathcal{F}$  and  $A \cap B \approx \emptyset$ . From the first premise we get  $A \cap B \notin \mathcal{F} \rightarrow \neg(A \in \mathcal{F} \& B \in \mathcal{F})$  by property 6. Since  $\emptyset \notin \mathcal{F}$  and  $A \cap B \approx \emptyset$ , we have  $A \cap B \notin \mathcal{F}$  and the result follows.
8. The formula  $A \cap B \approx \emptyset \rightarrow \neg(A \in \mathcal{F} \& B \in \mathcal{F})$  was derived under the same premises in the last proof. The conclusion follows from the formula  $\neg(\varphi \& \psi) \rightarrow (\neg\varphi \vee \neg\psi)$  provable in LII.
9. The last formula together with  $A \cap \setminus A = \emptyset$ .
10. The premises are  $\text{IntClosed}(\mathcal{F})$ ,  $\emptyset \notin \mathcal{F}$ ,  $A \cap B \approx \emptyset$  and  $A \in \mathcal{F}$ . We get  $A \notin \mathcal{F} \vee B \notin \mathcal{F}$  by using the first three premises with property 8. Further, from  $A \in \mathcal{F}$  combined with  $A \notin \mathcal{F} \vee B \notin \mathcal{F}$  we can derive  $B \notin \mathcal{F}$  (as the formula  $\varphi \& (\neg\varphi \vee \neg\psi) \rightarrow \neg\psi$  is provable in LII).



11. Follows straightforwardly from the previous one.

QED

**Definition 3.2** *We define unary predicate Filter as follows:*

$$\text{Filter}(\mathcal{F}) \equiv_{\text{df}} V \in \mathcal{F} \ \& \ \emptyset \notin \mathcal{F} \ \& \ \text{Mon}(\mathcal{F}) \ \& \ \text{IntClosed}(\mathcal{F})$$

Let us emphasize that the definition is graded: a fuzzy class  $\mathcal{F}$  is a filter to a degree to which the multiple conjunction in Definition 3.2 is satisfied. We remark that the kind of definition above is in spirit of the methodology proposed in [12] by Höhle: a fuzzy counterpart of some classical definition is obtained by reinterpreting the classical definition within fuzzy logic. “Gradedness” of filter makes our approach more general than the more traditional ones; vast majority of the existing concepts of filters can be captured by some variant of the formula

$$V \in \mathcal{F} \ \& \ \Delta(\emptyset \notin \mathcal{F}) \ \& \ \Delta \text{Mon}(\mathcal{F}) \ \& \ \Delta \text{IntClosed}(\mathcal{F})$$

in which all the subformulas except  $V \in \mathcal{F}$  must be satisfied to degree one. Accordingly, filters to degree one are described as  $\Delta \text{Filter}(\mathcal{F})$ .

Definition 3.2 can be relativized with respect to any set of connectives definable in logic LII. For example, in case of Gödel connectives, we put:

$$\begin{aligned} \text{Mon}_G(\mathcal{F}) &\equiv_{\text{df}} (\forall A, B)((A \in \mathcal{F} \wedge A \sqsubseteq B) \rightarrow_G B \in \mathcal{F}) \\ \text{IntClosed}_G(\mathcal{F}) &\equiv_{\text{df}} (\forall A, B)((A \in \mathcal{F} \wedge B \in \mathcal{F}) \rightarrow_G A \sqcap B \in \mathcal{F}) \end{aligned}$$

A notion of *Gödel filter* (to a certain degree, of course) is then captured by the predicate

$$\text{Filter}_G(\mathcal{F}) \equiv_{\text{df}} V \in \mathcal{F} \wedge \emptyset \notin_P \mathcal{F} \wedge \text{Mon}_G(\mathcal{F}) \wedge \text{IntClosed}_G(\mathcal{F})$$

Analogously, unary predicates  $\text{Filter}_P, \text{Filter}_L$  are introduced with respect to product (Łukasiewicz) connectives meaning that  $\mathcal{F}$  is *product (Łukasiewicz) filter*. The next proposition states (expected) properties of filters.

**Proposition 3.3 (Properties of filters)** *The following formulas are provable:*

1.  $\text{Filter}(\mathcal{F}) \rightarrow (A \sqcap B \in \mathcal{F} \rightarrow (A \in \mathcal{F} \wedge B \in \mathcal{F}))$

2.  $(\text{Filter}(\mathcal{F}) \ \& \ \text{Mon}(\mathcal{F})) \rightarrow ((A \cap B \in \mathcal{F})^2 \rightarrow (A \in \mathcal{F} \ \& \ B \in \mathcal{F}))$
3.  $(\text{Filter}(\mathcal{F}) \ \& \ \text{Mon}(\mathcal{F})) \rightarrow ((A \in \mathcal{F} \vee B \in \mathcal{F}) \rightarrow A \cup B \in \mathcal{F})$
4.  $\text{Filter}(\mathcal{F}) \rightarrow (B \notin \mathcal{F} \rightarrow (A \notin \mathcal{F} \vee A \not\subseteq B))$
5.  $\text{Filter}(\mathcal{F}) \rightarrow ((B \notin \mathcal{F} \ \& \ A \subseteq B) \rightarrow A \notin \mathcal{F})$
6.  $\text{Filter}(\mathcal{F}) \rightarrow (A \cap B \notin \mathcal{F} \rightarrow \neg(A \in \mathcal{F} \ \& \ B \in \mathcal{F}))$
7.  $\text{Filter}(\mathcal{F}) \rightarrow (A \cap B \approx \emptyset \rightarrow \neg(A \in \mathcal{F} \ \& \ B \in \mathcal{F}))$
8.  $\text{Filter}(\mathcal{F}) \rightarrow (A \cap B \approx \emptyset \rightarrow (A \notin \mathcal{F} \vee B \notin \mathcal{F}))$
9.  $\text{Filter}(\mathcal{F}) \rightarrow (A \notin \mathcal{F} \vee \setminus A \notin \mathcal{F})$
10.  $\text{Filter}(\mathcal{F}) \rightarrow (A \cap B \approx \emptyset \rightarrow (A \in \mathcal{F} \rightarrow B \notin \mathcal{F}))$
11.  $\text{Filter}(\mathcal{F}) \rightarrow (A \in \mathcal{F} \rightarrow \setminus A \notin \mathcal{F})$

**Proof:** Each formula is a straightforward consequence of the corresponding property from Lemma 3.1. QED

Finite Intersection Property is related to the behavior of fuzzy classes of fuzzy classes with respect to a finite application of intersection. We introduce the corresponding predicate in the next definition.

**Definition 3.4** *For every natural number  $n$  we define:*

$$\text{FIP}_n(\mathcal{F}) \equiv_{\text{df}} (\forall A_1, \dots, A_n)((A_1 \in \mathcal{F} \ \& \ \dots \ \& \ A_n \in \mathcal{F}) \rightarrow A_1 \cap \dots \cap A_n \not\approx \emptyset)$$

Broadly speaking, the next proposition says that every fuzzy class of fuzzy classes which is closed with respect to intersection to a “sufficiently high” truth degree satisfies Finite Intersection Property for every natural  $n$ .

**Proposition 3.5** *It is provable for every natural number  $n \geq 2$ :*

1.  $(\emptyset \notin \mathcal{F} \ \& \ \text{IntClosed}^{n-1}(\mathcal{F})) \rightarrow \text{FIP}_n(\mathcal{F})$
2.  $(\text{Filter}(\mathcal{F}) \ \& \ \text{IntClosed}^{n-2}(\mathcal{F})) \rightarrow \text{FIP}_n(\mathcal{F})$
3.  $\Delta \text{Filter}(\mathcal{F}) \rightarrow \text{FIP}_n(\mathcal{F})$

**Proof:**

1. From the premises  $\text{IntClosed}^{n-1}(\mathcal{F})$  and  $A_1 \in \mathcal{F}, \dots, A_n \in \mathcal{F}$ , we derive  $A_1 \cap \dots \cap A_n \in \mathcal{F}$  by using  $\text{IntClosed}(\mathcal{F})$  exactly  $(n - 1)$ -times. Combining this formula with the assumption  $\emptyset \notin \mathcal{F}$ , we can conclude that  $A_1 \cap \dots \cap A_n \not\approx \emptyset$ .
2. Follows immediately from the preceding case.
3. Follows from the previous one.

QED

In the rest of this section we investigate filters with respect to the three basic t-norms.

**Proposition 3.6 (Gödel filters)** *The following formulas are provable:*

1.  $\text{Filter}_G(\mathcal{F}) \rightarrow ((A \in \mathcal{F} \wedge B \in \mathcal{F}) \leftrightarrow_G A \sqcap B \in \mathcal{F})$
2.  $\text{Filter}_G(\mathcal{F}) \rightarrow ((A \in \mathcal{F} \vee B \in \mathcal{F}) \rightarrow_G A \sqcup B \in \mathcal{F})$

**Proof:**

1. Follows from  $\text{IntClosed}(\mathcal{F})$  together with property 1. from Proposition 3.3.
2. Follows from property 3. (Proposition 3.3).

QED

In the following proposition we shall see that product filters behave in a similar way as Gödel filters concerning strict negation and complement.

**Proposition 3.7 (Product filters)** *The following formulas are provable:*

1.  $\text{Filter}_P(\mathcal{F}) \rightarrow (A \cap_P B \approx_P \emptyset \rightarrow_P (A \notin_P \mathcal{F} \vee B \notin_P \mathcal{F}))$
2.  $\text{Filter}_P(\mathcal{F}) \rightarrow (A \notin_P \mathcal{F} \vee \setminus_P A \notin_P \mathcal{F})$

**Proof:**

1. Property 8. from Proposition 3.3 implies

$$A \cap_P B \approx_P \emptyset \rightarrow_P (A \notin_P \mathcal{F} \vee_P B \notin_P \mathcal{F})$$

where the succedent is equivalent to

$$A \notin_P \mathcal{F} \vee B \notin_P \mathcal{F}$$

since the formula  $(\neg_P \varphi \vee_P \neg_P \psi) \leftrightarrow (\neg_P \varphi \vee \neg_P \psi)$  is provable in product logic.

2. Follows immediately from the previous one.

QED

The point 1. of the following proposition reveals an interesting property of Łukasiewicz filters: every fuzzy class  $A$  is high (in the sense of predicate Height — see Definition 2.7) at least to the degree to which  $A$  belongs to the Łukasiewicz filter  $\mathcal{F}$ . We shall show later in Corollary 3.15 that this connection between degrees of membership  $A \in \mathcal{F}$  and  $x \in A$  is preserved neither in case of Gödel nor product filters.

**Proposition 3.8 (Łukasiewicz filters)** *The following formulas are provable:*

1.  $\text{Filter}_L(\mathcal{F}) \rightarrow (A \in \mathcal{F} \rightarrow \text{Height}(A))$
2.  $\text{Filter}_L(\mathcal{F}) \rightarrow (\backslash_L A \in \mathcal{F} \rightarrow A \notin_L \mathcal{F})$

**Proof:**

1. The assumptions are  $\text{Filter}_L(\mathcal{F})$  and  $A \in \mathcal{F}$ . We get the formula  $A \in \mathcal{F} \rightarrow (A \subseteq \emptyset \rightarrow_L \emptyset \in \mathcal{F})$  from  $\text{Mon}_L$  and residuation. Consequently, we have  $\emptyset \notin_L \mathcal{F} \rightarrow \neg_L(A \subseteq \emptyset)$  and, applying the assumption  $\emptyset \notin_L \mathcal{F}$  and rewriting the succedent, we get  $\neg_L(\forall x)(x \in A \rightarrow 0)$ . The last formula is in Łukasiewicz logic equivalent to the formula  $(\exists x)(x \in A)$  and thus  $\text{Height}(A)$ .
2. The result follows directly from 1. (Proposition 3.3) and properties of involutive negation.

QED

## 3.2 Base

Base and filter base are fuzzy classes of “generators” for a filter. We introduce a definition for an arbitrary fuzzy class  $\mathcal{F}$  at the beginning.

**Definition 3.9** *We define:*

$$\begin{aligned} \text{Base}(\mathcal{B}, \mathcal{F}) &\equiv_{\text{df}} (\forall A)(A \in \mathcal{F} \leftrightarrow (\exists B)(B \in \mathcal{B} \ \& \ B \subseteq A)) \\ \text{FilterBase}(\mathcal{B}) &\equiv_{\text{df}} \emptyset \notin \mathcal{B} \ \& \ (\exists A)(A \in \mathcal{B}) \ \& \\ &\quad \& \ (\forall A, B)((A \in \mathcal{B} \ \& \ B \in \mathcal{B}) \rightarrow (\exists C)(C \in \mathcal{B} \ \& \ C \subseteq A \cap B)) \end{aligned}$$

The intended meaning of formula  $\text{Base}(\mathcal{B}, \mathcal{F})$  is that “ $\mathcal{B}$  is a base for  $\mathcal{F}$ ”. In the next proposition we prove that every base for a fuzzy class is contained in that fuzzy class,<sup>1</sup> every filter is its own filter base, and a useful characterization of any filter base.

**Proposition 3.10 (Properties of base)** *It is provable:*

1.  $\text{Base}(\mathcal{B}, \mathcal{F}) \rightarrow \mathcal{B} \subseteq \mathcal{F}$
2.  $\text{Filter}(\mathcal{F}) \rightarrow \text{FilterBase}(\mathcal{F})$
3.  $\text{Filter}(\mathcal{F}) \ \& \ \Delta \text{Base}(\mathcal{B}, \mathcal{F}) \rightarrow \text{FilterBase}(\mathcal{B})$
4.  $\text{Filter}(\mathcal{F}) \ \& \ \text{Base}^5(\mathcal{B}, \mathcal{F}) \rightarrow \text{FilterBase}(\mathcal{B})$
5.  $\text{FilterBase}(\mathcal{B}) \rightarrow (\exists \mathcal{F})(\text{Filter}(\mathcal{F}) \ \& \ \text{Base}(\mathcal{B}, \mathcal{F}))$

**Proof:**

1. The premise is  $\text{Base}(\mathcal{B}, \mathcal{F})$  and  $A \in \mathcal{B}$ . Hence  $(\exists B)(B \in \mathcal{B} \ \& \ B \subseteq A)$  which is  $A \in \mathcal{F}$ .
2. Straightforwardly follows from the premise.
3. Since  $\emptyset \notin \mathcal{F}$  and  $\mathcal{B} \subseteq \mathcal{F}$  (property 1.), we have  $\emptyset \notin \mathcal{F} \rightarrow \emptyset \notin \mathcal{B}$ , thus  $\emptyset \notin \mathcal{B}$ . Next, the premise  $V \in \mathcal{F}$  implies  $(\exists B)(B \in \mathcal{B} \ \& \ B \subseteq V)$  which is precisely  $(\exists B)(B \in \mathcal{B})$ . Finally, we assume  $A \in \mathcal{B} \ \& \ B \in \mathcal{B}$  and hence  $A \in \mathcal{F} \ \& \ B \in \mathcal{F}$  (property 1.). From  $\text{IntClosed}(\mathcal{F})$  we get  $A \cap B \in \mathcal{F}$  which is the same as  $(\exists C)(C \in \mathcal{B} \ \& \ C \subseteq A \cap B)$ .

---

<sup>1</sup>In a pedantic manner, the aforementioned implication says that “ $\mathcal{B}$  is contained in  $\mathcal{F}$  at least to a degree to which  $\mathcal{B}$  is a base for  $\mathcal{F}$ ”.

4. Checking all the steps of the previous proof, we find out that the premise  $\text{Base}(\mathcal{F})$  was employed precisely 5-times.
5. We have to find a filter  $\mathcal{F}$  whose base is  $\mathcal{B}$ . For this purpose, we define

$$\mathcal{F} =_{\text{df}} \{A \mid (\exists B)(B \in \mathcal{B} \ \& \ B \subseteq A)\}$$

and it follows directly from this definition that  $\text{Base}(\mathcal{B}, \mathcal{F})$ . Further, we have  $\emptyset \notin \mathcal{F} \leftrightarrow (\forall B)\neg(B \in \mathcal{B} \ \& \ B \subseteq \emptyset)$ . Since the assumption is  $\emptyset \notin \mathcal{B}$ , we get  $\emptyset \notin \mathcal{F}$  by the definition of  $\text{Base}$ . Similarly,  $V \in \mathcal{F} \leftrightarrow (\exists B)(B \in \mathcal{B} \ \& \ B \subseteq V)$  which gives  $V \in \mathcal{F} \leftrightarrow (\exists B)(B \in \mathcal{B})$  and thus  $V \in \mathcal{F}$ . To prove  $\text{Mon}(\mathcal{F})$ , we assume  $A \in \mathcal{F}$  and  $A \subseteq B$ . The first assumption means  $(\exists C)(C \in \mathcal{B} \ \& \ C \subseteq A)$  and the second assumption now gives  $(\exists C)(C \in \mathcal{B} \ \& \ C \subseteq B)$ , hence  $B \in \mathcal{F}$ . Finally, we are going to show  $\text{IntClosed}(\mathcal{F})$ . The premise is  $A \in \mathcal{F} \ \& \ B \in \mathcal{F}$ . Hence

$$(\exists C)(C \in \mathcal{B} \ \& \ C \subseteq A) \ \& \ (\exists D)(D \in \mathcal{B} \ \& \ D \subseteq B)$$

from which we infer

$$(\exists C, D)(C \in \mathcal{B} \ \& \ D \in \mathcal{B} \ \& \ C \subseteq A \ \& \ D \subseteq B)$$

Further, employing the last formula from the definition of  $\text{FilterBase}(\mathcal{B})$ , we obtain

$$(\exists C, D, E)(E \in \mathcal{B} \ \& \ E \subseteq C \cap D \ \& \ C \subseteq A \ \& \ D \subseteq B)$$

Thus  $(\exists E)(E \in \mathcal{B} \ \& \ E \subseteq A \cap B)$  and the conclusion  $A \cap B \in \mathcal{F}$  follows.

QED

We will present a non-trivial example of filter base. A (crisp) class  $\text{Supp}(A)$  defined below stands for a *support* of the fuzzy class  $A$ , that is, all elements of the universe belonging to  $A$  with a non-zero degree of membership.

**Definition 3.11** *We define:*

$$\begin{aligned} \text{Supp}(A) &=_{\text{df}} \setminus_P \setminus_P A \\ \mathcal{B}(C) &=_{\text{df}} \{A \mid A \subseteq C \ \& \ \text{Supp}(A) = \text{Supp}(C)\} \end{aligned}$$

**Proposition 3.12** *It is provable:*

1.  $C \neq \emptyset \rightarrow \text{FilterBase}_G(\mathcal{B}(C))$
2.  $C \neq \emptyset \rightarrow \text{FilterBase}_P(\mathcal{B}(C))$

**Proof:** Both proofs are carried out jointly: in what follows, all operations and connectives are relativized with respect to either Gödel or product t-norm. We have  $\emptyset \notin \mathcal{B}(C) \leftrightarrow \neg(\emptyset \subseteq C \ \& \ \text{Supp}(C) = \emptyset)$  and further, also  $C \in \mathcal{B}(C) \leftrightarrow (C \subseteq C \ \& \ \text{Supp}(C) = \text{Supp}(C))$ , thus  $(\exists C)(C \in \mathcal{B}(C))$ . Assume  $D_1 \in \mathcal{B}(C)$ ,  $D_2 \in \mathcal{B}(C)$ . We obtain from Definition 3.11 the formulas

$$\begin{aligned} D_1 \subseteq C \quad &\& \quad \text{Supp}(D_1) = \text{Supp}(C) \\ D_2 \subseteq C \quad &\& \quad \text{Supp}(D_2) = \text{Supp}(C) \end{aligned}$$

Clearly,  $D_1 \cap D_2 \subseteq D_1 \cap D_2$  and  $D_1 \cap D_2 \subseteq C$ . Further, we have

$$\begin{aligned} \text{Supp}(D_1 \cap D_2) &= \setminus_P \setminus_P (D_1 \cap D_2) = \setminus_P (\setminus_P D_1 \cup \setminus_P D_2) \\ &= \setminus_P \setminus_P D_1 \cap \setminus_P \setminus_P D_2 = \text{Supp}(C) \end{aligned}$$

since  $\cap$  is an intersection corresponding to Gödel (product) t-norm. Hence  $(\exists D)(D \in \mathcal{B}(C) \ \& \ D \subseteq D_1 \cap D_2)$ . QED

The so called *characteristic value* of filter base was introduced in [4] as a supremum over all membership degrees of sets in that filter base. In our formal setting, this concept corresponds to the predicate  $\text{Height}(\mathcal{F})$  expressing height (see Definition 2.7) of the filter base  $\mathcal{F}$ . In case that  $\mathcal{F}$  is even a filter, observe provability of the following formula:

$$\text{Filter}(\mathcal{F}) \rightarrow (\text{Height}(\mathcal{F}) \leftrightarrow V \in \mathcal{F})$$

### 3.3 Constructions

In this section our intention is to show how filters arise in a constructive way and which operations preserve a property of fuzzy class “being a filter”.

**Definition 3.13** *We put:*

$$\begin{aligned} \mathcal{F}_x &=_{\text{df}} \{A \mid x \in A\} \\ \mathcal{F}_C &=_{\text{df}} \{A \mid C \subseteq A\} \\ \mathcal{F}_B &=_{\text{df}} \{A \mid (\exists B)(B \in \mathcal{B} \ \& \ B \subseteq A)\} \end{aligned}$$

If  $C = \{x\}$ , then  $\mathcal{F}_x$  is the same as  $\mathcal{F}_{\{x\}}$ . In addition, when  $\{C\}$  is a crisp class containing a single element  $C$ , observe provability of the formula  $\text{Base}(\{C\}, \mathcal{F}_C)$ . Similarly, notice that  $\text{Base}(\mathcal{B}, \mathcal{F}_B)$ .

**Proposition 3.14** *The following formulas are provable:*

1.  $\text{Filter}(\mathcal{F}_x)$
2.  $(C \neq \emptyset \ \& \ \text{Crisp}(C)) \rightarrow \text{Filter}(\mathcal{F}_C)$
3.  $C \neq \emptyset \rightarrow \text{Filter}_G(\mathcal{F}_C)$
4.  $C \neq \emptyset \rightarrow \text{Filter}_G(\mathcal{F}_{\mathcal{B}(C)})$
5.  $C \neq \emptyset \rightarrow \text{Filter}_P(\mathcal{F}_{\mathcal{B}(C)})$

**Proof:**

1. We have  $x \in V \leftrightarrow V \in \mathcal{F}_x$  and  $x \notin \emptyset \leftrightarrow \emptyset \notin \mathcal{F}_x$ , hence  $V \in \mathcal{F}_x$  and  $\emptyset \notin \mathcal{F}_x$ . To prove  $\text{Mon}(\mathcal{F}_x)$ , assume  $A \in \mathcal{F}_x$  and  $A \subseteq B$ . We get  $x \in A$  and thus  $x \in B$  which is the same as  $B \in \mathcal{F}_x$ . To show  $\text{IntClosed}(\mathcal{F}_x)$ , assume  $A \in \mathcal{F}_x \ \& \ B \in \mathcal{F}_x$ . We get  $x \in A \ \& \ x \in B$  which means  $x \in A \cap B$  and hence  $A \cap B \in \mathcal{F}_x$ .
2. We get  $V \in \mathcal{F}_C$  and  $\emptyset \notin \mathcal{F}_C$ . If  $A \in \mathcal{F}_C$  and  $A \subseteq B$ , then  $C \subseteq A \subseteq B$ . Hence  $B \in \mathcal{F}_C$ . Finally, from  $A \in \mathcal{F}_C \ \& \ B \in \mathcal{F}_C$  we obtain the formula  $C \subseteq A \ \& \ C \subseteq B$  which leads to  $C \subseteq A \cap B$  (since  $C$  is crisp) and thus  $A \cap B \in \mathcal{F}_C$ .
3. We get  $V \in \mathcal{F}_C$  and  $\emptyset \notin \mathcal{F}_C$ . If  $A \in \mathcal{F}_C$  and  $A \sqsubseteq B$ , then  $C \sqsubseteq A \sqsubseteq B$ . Hence  $B \in \mathcal{F}_C$ . Finally, from  $A \in \mathcal{F}_C \ \& \ B \in \mathcal{F}_C$  we obtain the formula  $C \sqsubseteq A \ \& \ C \sqsubseteq B$  which gives  $C \sqsubseteq A \sqcap B$  and thus  $A \sqcap B \in \mathcal{F}_C$ .
4. Recalling the notation introduced in Definition 3.11, we use Proposition 3.12 to derive  $\text{FilterBase}(\mathcal{B}(C))$ . Finally, taking into account a construction of the filter from the proof of property 5.(Proposition 3.10), we get the desired conclusion.
5. The same line of reasoning as in the previous case.

QED



Among others, the last proposition reveals difficulties we encounter when defining filter generated by a given fuzzy class  $C$ : idempotence of min-conjunction is the key property here.

The next corollary to Proposition 3.14 shows that there exist Gödel (product) filters in which an arbitrarily “small” non-empty fuzzy class (in the sense of predicate Height) belongs with degree one — compare with Proposition 3.8 expressing quite the opposite property of every Łukasiewicz filter.

**Corollary 3.15** *It is provable:*

1.  $C \neq \emptyset \rightarrow (\text{Filter}_G(\mathcal{F}_C) \ \& \ \Delta(C \in \mathcal{F}_C))$
2.  $C \neq \emptyset \rightarrow (\text{Filter}_P(\mathcal{F}_{B(C)}) \ \& \ \Delta(C \in \mathcal{F}_{B(C)}))$

Next, we shall deal with unions and intersections of filters. We distinguish usual union (intersection) of two filters ( $\mathcal{F} \cup \mathcal{G} = \{A \mid A \in \mathcal{F} \ \vee \ A \in \mathcal{G}\}$  and  $\mathcal{F} \cap \mathcal{G} = \{A \mid A \in \mathcal{F} \ \& \ A \in \mathcal{G}\}$ ) and union (intersection) of an arbitrary class of filters (see Definition 2.6). At first, we formulate a lemma.

**Lemma 3.16** *It is provable:*

1.  $(\text{Mon}_G(\mathcal{F}) \ \wedge \ \text{Mon}_G(\mathcal{G})) \rightarrow \text{Mon}_G(\mathcal{F} \cap \mathcal{G})$
2.  $(\Delta \text{Mon}(\mathcal{F}) \ \& \ \Delta \text{Mon}(\mathcal{G})) \rightarrow \Delta \text{Mon}(\mathcal{F} \cap \mathcal{G})$
3.  $(\text{IntClosed}(\mathcal{F}) \ \& \ \text{IntClosed}(\mathcal{G})) \rightarrow \text{IntClosed}(\mathcal{F} \cap \mathcal{G})$

**Proof:**

1. The premises are

$$(A \in \mathcal{F} \ \wedge \ A \in \mathcal{G} \ \wedge \ A \subseteq B \ \wedge \ A \subseteq B) \rightarrow (B \in \mathcal{F} \ \wedge \ B \in \mathcal{G})$$

and  $A \in \mathcal{F} \cap \mathcal{G} \ \wedge \ A \subseteq B$ . Thus the conclusion follows immediately by idempotence of min-conjunction and modus ponens.

2. Rewriting the premises, we get

$$(\Delta(A \in \mathcal{F}) \ \& \ \Delta(A \in \mathcal{G}) \ \& \ \Delta(A \subseteq B) \ \& \ \Delta(A \subseteq B)) \rightarrow (\Delta B \in \mathcal{F} \ \& \ \Delta B \in \mathcal{G})$$

and  $\Delta(A \in \mathcal{F} \cap \mathcal{G}) \ \& \ \Delta(A \subseteq B)$ . Hence  $\Delta(B \in \mathcal{F} \cap \mathcal{G})$  by modus ponens.

3. From the assumptions

$$(A \in \mathcal{F} \ \& \ A \in \mathcal{G} \ \& \ B \in \mathcal{F} \ \& \ B \in \mathcal{G}) \rightarrow (A \cap B \in \mathcal{F} \ \& \ A \cap B \in \mathcal{G})$$

and  $A \in \mathcal{F} \cap \mathcal{G} \ \& \ B \in \mathcal{F} \cap \mathcal{G}$  we get the result by modus ponens.

QED

**Proposition 3.17 (Intersection and union of two filters)** *It is provable:*

1.  $(\text{Filter}_G(\mathcal{F}) \ \& \ \text{Filter}_G(\mathcal{G})) \rightarrow \text{Filter}_G(\mathcal{F} \cap \mathcal{G})$
2.  $(\Delta \text{Filter}(\mathcal{F}) \ \& \ \Delta \text{Filter}(\mathcal{G})) \rightarrow \Delta \text{Filter}(\mathcal{F} \cap \mathcal{G})$
3.  $(\text{Filter}(\mathcal{F}) \ \& \ \text{Filter}(\mathcal{G}) \ \& \ (\mathcal{F} \subseteq \mathcal{G} \vee \mathcal{G} \subseteq \mathcal{F})) \rightarrow \text{Filter}(\mathcal{F} \sqcup \mathcal{G})$

**Proof:**

1. Rewriting both premises and using 1.,3. from Lemma 3.16, we get the result.
2. Similarly as in the preceding proof, we use 2., 3. from Lemma 3.16.
3. We have  $V \in \mathcal{F} \sqcup \mathcal{G}$  as  $V \in \mathcal{F} \ \& \ V \in \mathcal{G}$  is a premise. Analogously, we get  $\emptyset \notin \mathcal{F} \sqcup \mathcal{G}$ . To prove  $\text{Mon}(\mathcal{F} \sqcup \mathcal{G})$ , assume  $A \in \mathcal{F} \sqcup \mathcal{G}$  and  $A \subseteq B$ . Further, we can proceed by proof by cases for the formula  $A \in \mathcal{F} \vee A \in \mathcal{G}$ : assume  $A \in \mathcal{F}$ . The premise is  $\text{Mon}(\mathcal{F})$  and together with  $A \subseteq B$  we get  $B \in \mathcal{F}$  and thus  $B \in \mathcal{F} \sqcup \mathcal{G}$ . The second case  $A \in \mathcal{G}$  is completely analogous. Finally, assume  $A \in \mathcal{F} \sqcup \mathcal{G}$  and  $B \in \mathcal{F} \sqcup \mathcal{G}$ . The two possible cases are given as the premise  $\mathcal{F} \subseteq \mathcal{G} \vee \mathcal{G} \subseteq \mathcal{F}$ . When  $\mathcal{F} \subseteq \mathcal{G}$ , then  $A \in \mathcal{G}$  and  $B \in \mathcal{G}$ . Hence, by  $\text{IntClosed}(\mathcal{G})$ , we get  $A \cap B \in \mathcal{G}$  which gives  $A \cap B \in \mathcal{F} \sqcup \mathcal{G}$ . Similarly, we get the same conclusion also in the second case  $\mathcal{G} \subseteq \mathcal{F}$ .

QED

In the rest of this section we will investigate intersection and union of a given family of filters. In both assertions of the next proposition we have the premise

$$\mathbb{F} \subseteq \{\mathcal{F} \mid \Delta \text{Filter}(\mathcal{F})\} \ \& \ \mathbb{F} \neq \emptyset \ \& \ \text{Crisp}(\mathbb{F})$$

and we are thus dealing with a crisp class of filters to degree one. In addition, notice that in 2. of the next proposition the family  $\mathbb{F}$  is required to form a “chain” of filters.

**Proposition 3.18 (Intersection and union of family of filters)** *It is provable:*

1.  $\text{Filter}(\bigcap \mathbb{F})$
2.  $(\forall \mathcal{F}, \mathcal{G})(\mathcal{F} \in \mathbb{F} \wedge \mathcal{G} \in \mathbb{F}) \rightarrow (\mathcal{F} \subseteq \mathcal{G} \vee \mathcal{G} \subseteq \mathcal{F}) \rightarrow \text{Filter}(\bigcup \mathbb{F})$

**Proof:**

1. We have  $(\forall \mathcal{F})(\mathcal{F} \in \mathbb{F} \rightarrow V \in \mathcal{F})$  and  $(\forall \mathcal{F})(\mathcal{F} \in \mathbb{F} \rightarrow \emptyset \notin \mathcal{F})$ , which reads as  $V \in \bigcap \mathbb{F}$  and  $\emptyset \notin \bigcap \mathbb{F}$ , respectively. Let  $A \in \bigcap \mathbb{F}$  and  $A \subseteq B$ . Since  $(\forall \mathcal{F})(\mathcal{F} \in \mathbb{F} \rightarrow A \in \mathcal{F})$  and  $\Delta \text{Filter}(\mathcal{F})$ , we obtain the formula  $(\forall \mathcal{F})(\mathcal{F} \in \mathbb{F} \rightarrow B \in \mathcal{F})$  which is the same as  $B \in \bigcap \mathbb{F}$ . Finally, if  $A \in \bigcap \mathbb{F}$  and  $B \in \bigcap \mathbb{F}$ , then  $(\forall \mathcal{F})(\mathcal{F} \in \mathbb{F} \rightarrow A \cap B \in \mathcal{F})$  and thus  $A \cap B \in \bigcap \mathbb{F}$ .
2. We have  $V \in \bigcup \mathbb{F}$  and  $\emptyset \notin \bigcup \mathbb{F}$  since the following formulas are provable:  $(\exists \mathcal{F})(\mathcal{F} \in \mathbb{F} \ \& \ V \in \mathcal{F})$  and  $\neg(\exists \mathcal{F})(\mathcal{F} \in \mathbb{F} \ \& \ \emptyset \in \mathcal{F})$ . To prove  $\text{Mon}(\bigcup \mathbb{F})$ , assume  $A \in \bigcup \mathbb{F} \ \& \ A \subseteq B$ . The last formula can be rewritten as  $(\exists \mathcal{F})(\mathcal{F} \in \mathbb{F} \ \& \ A \in \mathcal{F} \ \& \ A \subseteq B)$  from which we get  $(\exists \mathcal{F})(\mathcal{F} \in \mathbb{F} \ \& \ B \in \mathcal{F})$ . To prove  $\text{IntClosed}(\bigcup \mathbb{F})$ , let us suppose  $A \in \bigcup \mathbb{F} \ \& \ B \in \bigcup \mathbb{F}$ . Hence

$$(\exists \mathcal{F})(\mathcal{F} \in \mathbb{F} \ \& \ A \in \mathcal{F}) \ \& \ (\exists \mathcal{G})(\mathcal{G} \in \mathbb{F} \ \& \ B \in \mathcal{G})$$

which gives  $(\exists \mathcal{F})(\mathcal{F} \in \mathbb{F} \ \& \ A \in \mathcal{F} \ \& \ (\exists \mathcal{G})(\mathcal{G} \in \mathbb{F} \ \& \ B \in \mathcal{G}))$ , and thus  $(\exists \mathcal{F}, \mathcal{G})(\mathcal{F} \in \mathbb{F} \ \& \ \mathcal{G} \in \mathbb{F} \ \& \ A \in \mathcal{F} \ \& \ B \in \mathcal{G})$ . Since  $\mathcal{F} \subseteq \mathcal{G} \vee \mathcal{G} \subseteq \mathcal{F}$  are the alternatives given as a premise, we can employ proof by cases. First, if  $\mathcal{F} \subseteq \mathcal{G}$ , then we obtain  $(\exists \mathcal{G})(\mathcal{G} \in \mathbb{F} \ \& \ A \in \mathcal{G} \ \& \ B \in \mathcal{G})$  which is  $(\exists \mathcal{G})(\mathcal{G} \in \mathbb{F} \ \& \ A \cap B \in \mathcal{G})$ . Second, if  $\mathcal{G} \subseteq \mathcal{F}$ , then we get  $(\exists \mathcal{F})(\mathcal{F} \in \mathbb{F} \ \& \ A \cap B \in \mathcal{F})$  analogously. In any case,  $A \cap B \in \bigcup \mathbb{F}$  and the proof is finished.

QED

In the last proposition (and its proof), notice that it is immaterial whether we use strong conjunction  $\&$  and disjunction  $\vee$  or min-conjunction  $\wedge$  and max-disjunction  $\vee$ , respectively, as the antecedent is a crisp formula.

### 3.4 Prime filter

**Definition 3.19** We define unary predicate  $\text{PrimeFilter}$  as

$$\text{PrimeFilter}(\mathcal{F}) \equiv_{\text{df}} \text{Filter}(\mathcal{F}) \ \& \ (\forall A, B)(A \cup B \in \mathcal{F} \leftrightarrow (A \in \mathcal{F} \vee B \in \mathcal{F}))$$

A similar definition (from the viewpoint of models, of course) appeared in [15, 5] for Gödel connectives. As usually, predicate  $\text{PrimeFilter}_G$  abbreviates the formula

$$\text{Filter}_G(\mathcal{F}) \wedge (\forall A, B)(A \sqcup B \in \mathcal{F} \leftrightarrow_G (A \in \mathcal{F} \vee B \in \mathcal{F}))$$

and predicates  $\text{PrimeFilter}_P, \text{PrimeFilter}_L$  have an analogous meaning with respect to the corresponding connectives.

**Proposition 3.20 (Properties of prime filters)** *It is provable:*

1.  $\text{PrimeFilter}_L(\mathcal{F}) \rightarrow (A \in \mathcal{F} \oplus \setminus_L A \in \mathcal{F})$
2.  $\text{PrimeFilter}_L(\mathcal{F}) \rightarrow (A \notin_L \mathcal{F} \leftrightarrow_L \setminus_L A \in \mathcal{F})$
3.  $\text{PrimeFilter}_L(\mathcal{F}) \rightarrow (A \in \mathcal{F} \leftrightarrow_L \setminus_L A \notin_L \mathcal{F})$
4.  $\Delta \text{PrimeFilter}_L(\mathcal{F}) \rightarrow (A \cap_L B \in \mathcal{F} \leftrightarrow_L (A \in \mathcal{F} \ \&_L \ B \in \mathcal{F}))$
5.  $\text{PrimeFilter}_G(\mathcal{F}) \rightarrow (\text{Crisp}(A) \rightarrow (A \in \mathcal{F} \vee A \notin_P \mathcal{F}))$
6.  $\text{PrimeFilter}_P(\mathcal{F}) \rightarrow (\text{Crisp}(A) \rightarrow (A \in \mathcal{F} \vee A \notin_P \mathcal{F}))$
7.  $\text{PrimeFilter}(\mathcal{F}_x)$

**Proof:**

1. The conclusion follows directly from the premise  $V \in \mathcal{F}$  and Definition 3.19 since

$$V \in \mathcal{F} \leftrightarrow_L (A \in \mathcal{F} \oplus \setminus_L A \in \mathcal{F})$$

2. The formulas  $A \in \mathcal{F} \oplus \setminus_L A \in \mathcal{F}$  and  $A \notin_L \mathcal{F} \rightarrow_L \setminus_L A \in \mathcal{F}$  are equivalent in Łukasiewicz logic. The rest is a consequence of 2. from Proposition 3.8.
3. Follows from the properties of involutive negation and equivalence.

4. We assume  $\Delta \text{PrimeFilter}(\mathcal{F})$  and  $A \cap_L B \in \mathcal{F}$ . We have

$$\backslash_L(A \cap_L B) = \backslash_L A \cup_L \backslash_L B$$

and thus  $\backslash_L A \cup_L \backslash_L B \notin_L \mathcal{F}$  by point 3. Using Definition 3.19, the last formula is equivalent to  $\backslash_L A \notin_L \mathcal{F} \&_L \backslash_L B \notin_L \mathcal{F}$  which reads as  $A \in \mathcal{F} \&_L B \in \mathcal{F}$  by 3. again.

5. Assumptions are  $\text{PrimeFilter}_G(\mathcal{F})$  and  $\text{Crisp}(A)$ . The first premise implies

$$A \notin_P \mathcal{F} \vee \backslash_P A \notin_P \mathcal{F} \quad (1)$$

(property 8. in Proposition 3.3) and

$$A \sqcup \backslash_P A \in \mathcal{F} \leftrightarrow_G (A \in \mathcal{F} \vee \backslash_P A \in \mathcal{F})$$

Due to the second premise, we obtain  $A \sqcup \backslash_P A = V$  and thus

$$A \in \mathcal{F} \vee \backslash_P A \in \mathcal{F} \quad (2)$$

Putting together (1) and (2), we conclude  $A \in \mathcal{F} \vee A \notin_P \mathcal{F}$ .

6. Assumptions are  $\text{PrimeFilter}_P(\mathcal{F})$  and  $\text{Crisp}(A)$ . The first premise implies

$$A \notin_P \mathcal{F} \underline{\vee}_P \backslash_P A \notin_P \mathcal{F} \quad (3)$$

(property 2. in Proposition 3.7) and, further,

$$A \cup_P \backslash_P A \in \mathcal{F} \leftrightarrow_P (A \in \mathcal{F} \underline{\vee}_P \backslash_P A \in \mathcal{F})$$

In product logic the formula from (3) is equivalent to

$$A \notin_P \mathcal{F} \vee \backslash_P A \notin_P \mathcal{F} \quad (4)$$

and the rest follows from the same line of reasoning as in the preceding proof.

7. Provability of  $\text{Filter}(\mathcal{F}_x)$  was already demonstrated in Proposition 3.14. It is thus enough to show  $A \cup B \in \mathcal{F}_x \leftrightarrow (A \in \mathcal{F}_x \underline{\vee} B \in \mathcal{F}_x)$ . However, this follows from the definition of  $\mathcal{F}_x$  as  $x \in A \cup B \leftrightarrow (x \in A \underline{\vee} x \in B)$ .

QED

It is worth emphasizing that only Łukasiewicz filters (concerning the three basic t-norms) preserve the “selective property” of classical prime filters (property 1. of the last proposition). Points 5. and 6. show the strong restriction on values of crisp sets belonging to a Gödel (product) prime filter: compare with the provable formula  $\text{Filter}_L \rightarrow (A \in \mathcal{F} \oplus A \notin_L \mathcal{F})$  in case of Łukasiewicz (prime) filters.<sup>2</sup>

Finally, we are going to demonstrate that every Łukasiewicz prime filter exhibits a certain maximality property.

**Proposition 3.21** *It is provable:*

$$\text{PrimeFilter}_L(\mathcal{F}) \rightarrow ((\text{Filter}_L(\mathcal{G}) \&_L \mathcal{F} \subseteq_L \mathcal{G}) \rightarrow_L \mathcal{G} \subseteq_L \mathcal{F})$$

**Proof:** The premises are

1.  $\text{PrimeFilter}_L(\mathcal{F})$
2.  $\text{Filter}_L(\mathcal{G})$
3.  $\mathcal{F} \subseteq_L \mathcal{G}$

We want to show  $\mathcal{G} \subseteq_L \mathcal{F}$ , hence we assume  $A \in \mathcal{G}$  in addition. Since  $A \in \mathcal{G} \rightarrow_L \backslash_L A \notin_L \mathcal{G}$  (premise 2. together with property 11. from Proposition 3.3), we get  $\backslash_L A \notin_L \mathcal{G}$ . As  $\backslash_L A \notin \mathcal{G}_L \rightarrow_L \backslash A \notin_L \mathcal{F}$  (premise 3.), we obtain  $\backslash_L A \notin_L \mathcal{F}$ . Modus ponens applied on the last formula and the formula  $\backslash_L A \notin_L \mathcal{F} \rightarrow_L A \in \mathcal{F}$  (premise 1. together with property 3. from Proposition 3.20) gives  $A \in \mathcal{F}$ .

QED

## 4 Models

Although the purely formal framework elaborated in Section 3 is more general as it essentially describes the graded properties of filters, we shall discuss some of the models below (mostly those corresponding to filters to degree one). First, because they appear in the existing literature about filters and, second, because those models represent mathematical objects which appears in measure and probability theory.

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<sup>2</sup>Clearly, the formula  $A \in \mathcal{F} \oplus A \notin_L \mathcal{F}$  is a tautology of Łukasiewicz logic.

In this paragraph, we view the real unit interval  $[0, 1]$  as the *standard LII algebra* [9]. When  $X$  is a non-empty set, then the product  $[0, 1]^X$  is an LII algebra endowed with pointwise operations (such as  $\neg$ ,  $\&$ ,  $\underline{\vee}$ ,  $\dots$ ) and pointwise ordering  $\leq$ . By  $\cap, \cup, \sqcap, \sqcup, \dots$  we denote the usual operations on fuzzy sets from Definition 2.5. Intended models of fuzzy classes on the first level are Zadeh-like fuzzy sets (elements of  $[0, 1]^X$ ) denoted as  $A, B, \dots$ . Filters are fuzzy classes on the second level. In the sequel, we will consider only filters described by the formula

$$\neg(\forall P \in \mathcal{F}) \& \Delta(\emptyset \notin \mathcal{F}) \& \Delta \text{Mon}(\mathcal{F}) \& \Delta \text{IntClosed}(\mathcal{F}) \quad (5)$$

in which all the subformulas except  $\forall P \in \mathcal{F}$  must be satisfied to degree one and the truth value of  $\forall P \in \mathcal{F}$  has to be non-zero. Semantically, models corresponding to formula (5) can be viewed as certain monotone functionals defined on fuzzy sets from  $[0, 1]^X$ . Namely, the models are functionals  $\mu : [0, 1]^X \rightarrow [0, 1]$  satisfying these conditions for every  $A, B \in [0, 1]^X$ :

1.  $\mu(X) = c, \quad c \in (0, 1]$ ,
2.  $\mu(\emptyset) = 0$ ,
3. if  $A \leq B$ , then  $\mu(A) \leq \mu(B)$ ,
4.  $\mu(A) \& \mu(B) \leq \mu(A \cap B)$ .

The characteristic value  $c$  (see the last paragraph of Section 3.2) corresponds to the truth degree of the formula  $\forall P \in \mathcal{F}$  and it is usually supposed that the whole universe truly belongs to the filter, that is,  $c = 1$ . The conditions 1.-4. above capture various concepts of filter existing in the literature:

- *L-filters* [14],
- *generalized filters* [4, 5] (in case  $\mu : \{0, 1\}^X \rightarrow [0, 1]$ ,  $\& = \wedge$ ,  $\cap = \sqcap$ ),
- *prefilters* [15, 16] (in case  $\mu : [0, 1]^X \rightarrow \{0, 1\}$ ,  $\& = \wedge$ ,  $\cap = \sqcap$ ).

We are convinced that the reader is able to work out definitions and basic results for predicates *Base*, *PrimeFilter* etc. in terms of models. We shall shortly comment on the interpretation of property 7. from Proposition 3.3:

$$\text{if } A \cap B = \emptyset, \text{ then } \mu(A) \& \mu(B) = 0.$$

Let us agree to call two elements  $x, y$  of an  $\mathbb{L}\Pi$ -algebra *disjoint* (with respect to a t-norm  $\&$  definable on the  $\mathbb{L}\Pi$  algebra) if  $a \& b = 0$ . It is then clear that every model of a filter  $\mu$  can be viewed as a “disjointness-preserving” map. Notice that this property severely restricts possible values of membership degrees of two disjoint fuzzy sets in the filter as  $a \& b = 0$  iff  $a = 0$  or  $b = 0$  unless the t-norm  $\&$  has *zero divisors*<sup>3</sup> (such as Łukasiewicz t-norm, for example). Further impact of this phenomenon is exhibited by many formulas listed in Proposition 3.3.

We are going to briefly discuss different models arising for the basic t-norms. Gödel filters correspond to *necessity measures on fuzzy sets* [20]. Models of Łukasiewicz filters include *probability measures on fuzzy sets* [18] and thus classical probabilities, too. Łukasiewicz prime filters give rise to a special class of the models which can be described as mappings preserving Łukasiewicz operations,<sup>4</sup> that is, for every  $A, B \in [0, 1^X]$ :

$$\begin{aligned}\mu(\setminus_L A) &= \neg_L \mu(A) \\ \mu(A \cup_L B) &= \mu(A) \oplus \mu(B) \\ \mu(A \cap_L B) &= \mu(A) \&_L \mu(B)\end{aligned}$$

It can be shown (see [17], for instance) that mappings satisfying the above conditions play a role of basic building blocks of probabilities on fuzzy sets. Therefore, we see that the well-known one-to-one correspondence between classical prime filters (or maximal filters, equivalently) and Boolean homomorphisms is preserved also in this more general setting.

## 5 Concluding Remarks

In the paper we have made first steps towards establishing a graded theory of fuzzy filters over fuzzy class theory. In conclusion let us mention several lines of research along which the presented approach can be further developed.

First, the fuzzy logic considered (logic  $\mathbb{L}\Pi$ ) can be replaced by a weaker logic while preserving provability of all theorems. For instance, a reasonably general yet sufficiently expressive alternative of  $\mathbb{L}\Pi$  would be the so called *monoidal t-norm based logic (MTL)* (see [8]). In this logic every left-continuous t-norms is definable and hence every filter could be relativized

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<sup>3</sup>We say that a t-norm  $\&$  has *zero divisors* if there exist  $a, b \in (0, 1)$  with  $a \& b = 0$ .

<sup>4</sup>Compare with Proposition 3.20.



with respect to such a t-norm. Further, both logics LPI and MTL enable to introduce a (*rational*) *truth constant*  $\bar{r}$  for every rational number  $r \in [0, 1]$ . In this way we can capture filter constructions depending directly on values from  $[0, 1]$  such as notions of stratified and tight filter [10].

Second, realize that the operations used in Definition 3.2 of filter are actually derived from one t-norm only. Therefore, in case of Gödel (product) filters we use strict complement and, on the other hand, involutive complement corresponds to Łukasiewicz filters. However, especially a use of involutive complement in place of strict one can fit desirable properties of fuzzy set complement in a more natural way. An analogous approach is adopted in the field of measures on fuzzy sets [18].

Third, a notion of prime filter known from classical theory is equivalent to that of maximal filter. Recall that a weaker version of this assertion relates to Łukasiewicz prime filters too (Proposition 3.21). Moreover, axiom of choice from classical theory (in the form of Zorn's lemma) makes possible to extend filters to maximal ones. There exist a version of axiom of choice for second-order logic as well — see [19] which advocates a role of second-order logic in mathematics — and we thus expect that adding a suitable variant of this axiom to fuzzy class theory can lead to an extension theorem for filters in our framework.

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